Relations of endograph metric and Γ-convergence on fuzzy sets ☆

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Abstract

This paper shows that the endograph metric and the Γ-convergence are compatible on a large class of fuzzy set in $\mathbb{R}^m$.

Keywords: Endograph metric; Γ-convergence; Hausdorff metric; compatible

1. Introduction

We show that the endograph metric and the Γ-convergence are compatible on a large class of fuzzy set in $\mathbb{R}^m$. The results in this paper improves the corresponding results in [5, 6]

2. Preliminaries

In this section, we recall and give some basic notions and fundamental results related to fuzzy sets and convergence structures on them. Readers can refer to [1–4] for related contents.

Throughout this paper, we suppose that X *is a nonempty set and* d *is the metric on* X. For simplicity, we also use X to denote the metric space (X, d).

The metric $\overline{d}$ on $X \times [0, 1]$ is defined as follows: for $(x, \alpha), (y, \beta) \in X \times [0, 1]$,

$$\overline{d}((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$

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Throughout this paper, we suppose that *the metric on* $X \times [0,1]$ *is* $\overline{d}$. For simplicity, we also use $X \times [0,1]$ to denote the metric space $(X \times [0,1], \overline{d})$.

A fuzzy set u in X can be seen as a function $u : X \to [0,1]$. A subset S of X can be seen as a fuzzy set in X. If there is no confusion, the fuzzy set corresponding to S is often denoted by χ_S; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use $\widehat{x}$ to denote the fuzzy set $\chi_{\{x\}}$ in X. In this paper, if we want to emphasize a specific metric space X, we will write the fuzzy set corresponding to S in X as $S_{F(X)}$, and the fuzzy set corresponding to $\{x\}$ in X as $\widehat{x}_{F(X)}$.

The symbol $F(X)$ is used to denote the set of all fuzzy sets in X. For $u \in F(X)$ and $\alpha \in [0,1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α*-cut* of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0,1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the topological closure of S in (X, d).

The symbol $K(X)$ and $C(X)$ are used to denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X, respectively.

Let $F_{USC}(X)$ denote the set of all upper semi-continuous fuzzy sets $u : X \to [0,1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \cup \{\emptyset\} \text{ for all } \alpha \in [0,1]\}.$$

Define

$$F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X) \cup \{\emptyset\}\},$$
$$F_{USCG}(X) := \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \cup \{\emptyset\} \text{ for all } \alpha \in (0,1]\}.$$

Clearly,

$$F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USC}(X).$$

Define

$$F_{CON}(X) := \{u \in F(X) : \text{for all } \alpha \in (0,1],\ [u]_\alpha \text{ is connected in } X\},$$

$$F_{USCCON}(X) := F_{USC}(X) \cap F_{CON}(X),$$
$$F_{USCGCON}(X) := F_{USCG}(X) \cap F_{CON}(X).$$

Let $u \in F_{CON}(X)$. Then $[u]_0 = \overline{\cup_{\alpha>0}[u]_\alpha}$ is connected in X.

If $u = \chi_\emptyset$, then $[u]_0 = \emptyset$ is connected in X. If $u \neq \chi_\emptyset$, then there is an $\alpha \in (0,1]$ such that $[u]_\alpha \neq \emptyset$. Note that $[u]_\beta \supseteq [u]_\alpha$ when $\beta \in [0,\alpha]$. Hence $\cup_{0<\beta<\alpha}[u]_\beta$ is connected, and thus $[u]_0 = \overline{\cup_{0<\beta<\alpha}[u]_\beta}$ is connected.

So

$$F_{CON}(X) = \{u \in F(X) : \text{for all } \alpha \in [0,1],\ [u]_\alpha \text{ is connected in } X\}.$$

Let $F^1_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0,1]$, i.e.,

$$F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0,1]\}.$$

We introduce some subclasses of $F^1_{USC}(X)$, which will be discussed in this paper. Define

$$F^1_{USCB}(X) := F^1_{USC}(X) \cap F_{USCB}(X),$$
$$F^1_{USCG}(X) := F^1_{USC}(X) \cap F_{USCG}(X),$$
$$F^1_{USCCON}(X) := F^1_{USC}(X) \cap F_{CON}(X),$$
$$F^1_{USCGCON}(X) := F^1_{USCG}(X) \cap F_{CON}(X).$$

Clearly,

$$F^1_{USCB}(X) \subseteq F^1_{USCG}(X) \subseteq F^1_{USC}(X),$$
$$F^1_{USCGCON}(X) \subseteq F^1_{USCCON}(X).$$

Let (X,d) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff distance*** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V),\ H^*(V,U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u\in U} d\,(u,V) = \sup_{u\in U}\inf_{v\in V} d\,(u,v).$$

If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0,1])$ induced by $\overline{d}$.

Remark 2.1. ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X, d), the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on $K(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$ is a metric. In these cases, we call the Hausdorff distance the Hausdorff metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but probably not a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X \times [0,1])$ induced by $\overline{d}$ is also not a metric. So the Hausdorff distance H on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$ is an extended metric but probably not a metric. In the cases that the Hausdorff distance H is an extended metric, we call the Hausdorff distance the Hausdorff extended metric.

We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m \times [0,1])$.

In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

For $u \in F(X)$, define

$$\operatorname{end} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\},$$
$$\operatorname{send} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\} \cap ([u]_0 \times [0,1]).$$

$\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph and the sendograph of u, respectively.

We can define the endograph metric H_{end} on $F_{USC}(X)$ as usual. For $u, v \in F_{USC}(X)$,

$$\boldsymbol{H_{\text{end}}(u,v)} := H(\operatorname{end} u, \operatorname{end} v),$$

where H is the Hausdorff metric on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$.

Rojas-Medar and Román-Flores [4] introduced the Kuratowski convergence of a sequence of sets in a metric space.

Let (X,d) be a metric space. Let C be a set in X and $\{C_n\}$ a sequence of sets in X. $\{C_n\}$ is said to ***Kuratowski converge*** to C according to (X,d), if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X : \ x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X : \ x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} C_m}.$$

In this case, we'll write $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ according to (X,d). If there is no confusion, we will not emphasize the metric space (X,d) and write $\{C_n\}$ ***Kuratowski converges*** to C or $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ for simplicity.

We can define the Γ-convergence of a sequence of fuzzy sets on $F_{USC}(X)$. as usual.

Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. $\{u_n\}$ is said to **$\boldsymbol{\Gamma}$-*converge*** to u, denoted by $\boldsymbol{u = \lim^{(\Gamma)}_{n\to\infty} u_n}$, if $\text{end}\, u = \lim^{(K)}_{n\to\infty} \text{end}\, u_n$ according to $(X \times [0,1], \overline{d})$.

3. Main results

We need the following conclusions.

Theorem 3.1. *[5] Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \ldots$. Then $H(C_n, C) \to 0$ implies that $\lim^{(K)}_{n\to\infty} C_n = C$.*

Lemma 3.2. *(Lemma 2.1 in [5]) Let (X,d) be a metric space, and C_n, $n = 1, 2, \ldots$, be a sequence of sets in X. Suppose that $x \in X$. Then*
(i) $x \in \liminf_{n\to\infty} C_n$ if and only if $\lim_{n\to\infty} d(x, C_n) = 0$,
(ii) $x \in \limsup_{n\to\infty} C_n$ if and only if there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\lim_{k\to\infty} d(x, C_{n_k}) = 0$.

Proof. Here we give a detailed proof. Readers who think that this conclusion is obvious can skip this proof.

(i) Assume that $x \in \liminf_{n\to\infty} C_n$. Then there is a sequence $\{x_n, n = 1, 2, \ldots\}$ in X such that $x_n \in C_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} d(x, x_n) = 0$. Since $d(x, C_n) \leq d(x, x_n)$, thus $\lim_{n\to\infty} d(x, C_n) = 0$.

Conversely, assume that $\lim_{n\to\infty} d(x, C_n) = 0$. For each $n = 1, 2, \ldots$, we can choose an x_n in C_n such that $d(x, x_n) \leq d(x, C_n) + 1/n$. Hence $\lim_{n\to\infty} d(x, x_n) = 0$. So $x \in \liminf_{n\to\infty} C_n$.

(ii) Assume that $x \in \limsup_{n\to\infty} C_n$. Then there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and $x_{n_k} \in C_{n_k}$ for $k = 1, 2, \ldots$ such that $\lim_{k\to\infty} d(x, x_{n_k}) = 0$. Since $d(x, C_{n_k}) \leq d(x, x_{n_k})$, thus $\lim_{k\to\infty} d(x, C_{n_k}) = 0$.

Conversely, assume that there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\lim_{k\to\infty} d(x, C_{n_k}) = 0$. For each $k = 1, 2, \ldots$, we can choose an x_{n_k} in C_{n_k} such that $d(x, x_{n_k}) \leq d(x, C_{n_k}) + 1/k$. Hence $\lim_{k\to\infty} d(x, x_{n_k}) = 0$. So $x \in \limsup_{n\to\infty} C_n$.

□

Theorem 3.3. *(Theorem 5.19 in [6]) Let u be a fuzzy set in $F^1_{USCG}(X)$ and let u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. Then the following are equivalent:*

(i) $H_{\mathrm{end}}(u_n, u) \to 0$*;*

(ii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *holds a.e. on* $\alpha \in (0, 1)$*;*

(iii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *for all* $\alpha \in (0, 1) \setminus P_0(u)$*;*

(iv) There is a dense subset P of $(0, 1)\backslash P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for $\alpha \in P$;

(v) There is a countable dense subset P of $(0, 1)\backslash P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for $\alpha \in P$.

Theorem 3.4. *(Theorem 6.2 in [5]) Let u, u_n, $n = 1, 2, \ldots$ be fuzzy sets in $F_{USC}(\mathbb{R}^m)$. Then the following are equivalent:*

(i) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$*;*

(ii) $\lim^{(K)}_{n\to\infty} [u_n]_\alpha = [u]_\alpha$ *holds a.e. on* $\alpha \in (0, 1)$*;*

(iii) $\lim^{(K)}_{n\to\infty} [u_n]_\alpha = [u]_\alpha$ *holds for all* $\alpha \in (0, 1) \setminus P(u)$*;*

(iv) There is a dense subset P of $(0, 1)\backslash P(u)$ such that $\lim^{(K)}_{n\to\infty} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$;

(v) There is a countable dense subset P of $(0, 1)\backslash P(u)$ such that $\lim^{(K)}_{n\to\infty} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$.

Proposition 3.5. *Let C be a nonempty compact set in $\mathbb{R}^m$ and for $n = 1, 2, \ldots$ let C_n be a nonempty connected and closed set in $\mathbb{R}^m$. Then $H(C_n, C) \to 0$ if and only if $\lim^{(K)}_{n\to\infty} C_n = C$.*

Proof. From Theorem 3.1, we have that $H(C_n, C) \to 0 \Rightarrow \lim^{(K)}_{n\to\infty} C_n = C$.

Now we show that $\lim^{(K)}_{n\to\infty} C_n = C \Rightarrow H(C_n, C) \to 0$. We prove by contradiction. Assume that $\lim^{(K)}_{n\to\infty} C_n = C$ but $H(C_n, C) \not\to 0$. Then $H^*(C, C_n) \not\to 0$ or $H^*(C_n, C) \not\to 0$. We split the proof into two cases.

Case (i) $H^*(C, C_n) \not\to 0$.

In this case, there is an $\varepsilon > 0$ and a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $H^*(C, C_{n_k}) > \varepsilon$. Thus for each $k = 1, 2, \ldots$ there exists $x_k \in C$ such that $d(x_k, C_{n_k}) > \varepsilon$. Since C is compact, there is a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ which converges to $x \in C$. Then there is a $L(\varepsilon)$ such that $d(x, x_{k_l}) < \varepsilon/2$ for all $l \geq L$. Hence $d(x, C_{n_{k_l}}) \geq d(x_{k_l}, C_{n_{k_l}}) - d(x, x_{k_l}) > \varepsilon/2$. By Lemma 3.2 (i), this contradicts $x \in C = \liminf_{n\to\infty} C_n$.

Case (ii) $H^*(C_n, C) \not\to 0$.

In this case, there is an $\varepsilon > 0$ and a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $H^*(C_{n_k}, C) > \varepsilon$. Thus we have the following:
(a) for each $k = 1, 2, \ldots$ there exists $x_{n_k} \in C_{n_k}$ such that $d(x_{n_k}, C) > \varepsilon$.

Pick $y \in C$, since $C = \liminf_{n\to\infty} C_n$, we can find a sequence $\{y_n\}$ satisfying that $y_n \in C_n$ for $n = 1, 2, \ldots$ and $\{y_n\}$ converges to y. Hence there is a $N(\varepsilon)$ such that for all $n \geq N$, $d(y_n, y) < \varepsilon$. Since $d(y_n, C) \leq d(y_n, y)$, we have the following:
(b) for all $n \geq N$, $d(y_n, C) < \varepsilon$.

Let $k \in \mathbb{N}$ with $n_k \geq N$. Define a function f_k from C_{n_k} to $\mathbb{R}$ given by $f_k(z) = d(z, C)$ for each $z \in C_{n_k}$. Then f_k is a continuous function on C_{n_k}. Since C_{n_k} is a connected set in $\mathbb{R}^m$, $f_k(C_{n_k})$ is a connected set in $\mathbb{R}$. Combined this fact with the above clauses (a) and (b), we obtain that there exists $z_{n_k} \in C_{n_k}$ such that

$$d(z_{n_k}, C) = \varepsilon. \tag{1}$$

From (1) and the compactness of C, the set $\{z_{n_k}, n_k \geq N\}$ is bounded in $\mathbb{R}^m$, and thus $\{z_{n_k}, n_k \geq N\}$ has a cluster point z. By (1), $d(z, C) = \varepsilon$, which contradicts $z \in \limsup_{n\to\infty} C_n = C$.

$\square$

Remark 3.6. Proposition 3.5 may be known, however we can't find this conclusion in the references that we can obtain. So we give a proof here.

Let A be a nonempty compact set in $\mathbb{R}^m$ and B a nonempty closed set in $\mathbb{R}^m$. If $H(A, B) < +\infty$, then B is bounded and hence a compact set in $\mathbb{R}^m$. Clearly, if B is compact in $\mathbb{R}^m$, then $H(A, B) < +\infty$. So $H(A, B) < +\infty$ if and only if B is a compact set in $\mathbb{R}^m$.

From the above fact we know that for C and C_n, $n = 1, 2, \ldots$, satisfying the assumptions of Proposition 3.5, if $H(C_n, C) \to 0$ (by Proposition 3.5 $H(C_n, C) \to 0$ if and only if $\lim^{(K)}_{n\to\infty} C_n = C$), then clearly there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $H(C_n, C) < +\infty$ and thus for all $n \geq N$, C_n is compact.

The following Proposition 3.7 is an immediate consequence of Proposition 3.5, Theorem 3.3, and Corollary 3.4.

Theorem 3.7. *Let u be a fuzzy set in $F^1_{USCG}(\mathbb{R}^m)$ and for $n = 1, 2, \ldots$, let u_n be a fuzzy set in $F^1_{USCCON}(\mathbb{R}^m)$. Then $H_{\rm end}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

Proof. The proof is routine.

By let $X = \mathbb{R}^m$ in Theorem 3.3 we have the following:

(i) $H_{\rm end}(u_n, u) \to 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$.

By Theorem 3.4, we have the following:

(ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ if and only if $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$ holds a.e. on $\alpha \in (0, 1)$.

Since for each $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, $[u]_\alpha \in K(\mathbb{R}^m)$, $[u_n]_\alpha \in C(\mathbb{R}^m)$ and $[u_n]_\alpha$ is connected in $\mathbb{R}^m$. Thus by Proposition 3.5, for each $\alpha \in (0, 1]$, $H([u_n]_\alpha, [u]_\alpha) \to 0$ if and only if $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$. Combined this fact with the above clauses (i) and (ii), we have that $H_{\rm end}(u_n, u) \to 0$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.

$\square$

Remark 3.8. Theorem 3.7 would be false if $\mathbb{R}^m$ were replaced by a general metric space X.

Here we mention that for $u \in F^1_{USCG}(\mathbb{R})$ and a sequence $\{u_n\}$ in $F^1_{USCCON}(\mathbb{R})$, $H(u_n, u) \to 0$ does not imply that there is an N satisfying that for all $n \geq N$ $u_n \in F^1_{USCGCON}(\mathbb{R})$.

The following Example 3.9 is a such example, which shows that there exists a $u \in F^1_{USCG}(\mathbb{R})$ and a sequence $\{u_n\}$ in $F^1_{USCCON}(\mathbb{R})$ such that

(i) $H(u_n, u) \to 0$,

(ii) for each $n = 1, 2, \ldots$, $u_n \notin F^1_{USCGCON}(\mathbb{R})$.

Example 3.9. Let $u = \widehat{1}_{F(\mathbb{R})} \in F^1_{UCCG}(\mathbb{R})$. For $n = 1, 2, \ldots$, define $u_n \in F^1_{USC}(\mathbb{R})$ as follows:

$$u_n(t) = \begin{cases} 1, & t = 1, \\ 1/n, & t \neq 1. \end{cases}$$

Then for $n = 1, 2, \ldots$,

$$[u_n]_\alpha = \begin{cases} \{1\}, & \alpha \in (1/n, 1], \\ \mathbb{R}, & \alpha \in [0, 1/n]. \end{cases}$$

So for each $n = 1, 2, \ldots$, $u_n \in F^1_{USCCON}(\mathbb{R})$ but $u_n \notin F^1_{USCGCON}(\mathbb{R})$. It can be seen that $H_{\rm end}(u, u_n) = 1/n \to 0$.

Theorem 9.2 in [5] discusses the compatibility of the endograph metric $H_{\rm end}$ and the Γ-convergence.

Theorem 3.10. *(Theorem 9.2 in [5]) Let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$ and for $n = 1, 2, \ldots$, let u_n be a fuzzy set in $F_{USCGCON}(\mathbb{R}^m)$. Then $H_{\rm end}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

The following Corollary 3.11 is an immediate corollary of Theorem 9.2 in [5].

Corollary 3.11. *Let u be a fuzzy set in $F^1_{USCG}(\mathbb{R}^m)$ and for $n = 1, 2, \ldots$, let u_n be a fuzzy set in $F^1_{USCGCON}(\mathbb{R}^m)$. Then $H_{\rm end}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

We can see that Theorem 3.7 is an improvement of Corollary 3.11.

Corollary 3.11 is the normal fuzzy set case of Theorem 9.2 in [5], which is an important special case of Theorem 9.2 in [5].

In the following, we give an improvement of Theorem 9.2 in [5].

For a subset S in $(X \times [0, 1], \overline{d})$, we still use $\overline{d}$ to denote the induced metric on S by $\overline{d}$.

For a set S in X, we use $\overline{S}$ to denote the topological closure of S in (X, d); for a set S in $X \times [0, 1]$, we use $\overline{S}$ to denote the topological closure of S in $(X \times [0, 1], \overline{d})$. The readers can judge the meaning of $\overline{S}$ according to the context.

For $D \subseteq X \times [0, 1]$ and $\alpha \in [0, 1]$, define $\boldsymbol{D_\alpha} := \{x \in X : (x, \alpha) \in D\}$. For a nonempty set D in $X \times [0, 1]$, define $\boldsymbol{f_D} := \inf\{\alpha : (x, \alpha) \in D\}$ and $\boldsymbol{S_D} := \sup\{\alpha : (x, \alpha) \in D\}$.

Proposition 3.12. *Let D be a nonempty set in $X \times [0,1]$ with $D_r \subseteq D_t$ for $f_D \leq t \leq r \leq 1$. Then D_{f_D} is connected in (X,d) if and only if D is connected in $(X \times [0,1], \overline{d})$.*

Proof. Sufficiency. We proceed by contradiction. Assume that D_{f_D} is connected in X. If D is not connected in $X \times [0,1]$, then there exists two nonempty sets A and B in $X \times [0,1]$ such that $A \cup B = D$, $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$.

Note that $D_{f_D} \times \{f_D\} \subseteq D$ and $D_{f_D} \times \{f_D\}$ is connected. Hence $D_{f_D} \times \{f_D\} \subseteq A$ or $D_{f_D} \times \{f_D\} \subseteq B$. Without loss of generality, we suppose that $D_{f_D} \times \{f_D\} \subseteq A$.

Pick $(x,\alpha) \in B$. Set $\gamma = \inf\{\beta : (x,\beta) \in B\}$.

If $(x,\gamma) \in B$, we affirm that $\gamma > f_D$. Otherwise $\gamma = f_D$ and $(x, f_D) \in B$. Note that $(x, f_D) \in A$. Thus $A \cap B \neq \emptyset$, which is a contradiction. Hence $(x,\xi) \in A$ for $\xi \in [f_D, \gamma)$, and therefore $(x,\gamma) = \lim_{\xi \to \gamma -}(x,\xi) \in \overline{A}$. So $\overline{A} \cap B \neq \emptyset$. This is a contradiction.

If $(x,\gamma) \in A$, then there is a sequence $\{(x,\gamma_n)\}$ in B such that $\lim_{n\to\infty} \gamma_n = \gamma$. Hence $(x,\gamma) = \lim_{n\to\infty}(x,\gamma_n) \in \overline{B}$. Thus $A \cap \overline{B} \neq \emptyset$. This is a contradiction.

Necessity. Assume that D is connected in $X \times [0,1]$. Define a function $f : (D, \overline{d}) \to (D_{f_D}, d)$ given by $f(x,\alpha) = x$ for each $(x,\alpha) \in D$. Clearly for $(x,\alpha), (y,\beta) \in X \times [0,1]$, $d(f(x,\alpha), f(y,\beta)) = d(x,y) \leq \overline{d}((x,\alpha),(y,\beta))$. Then f is continuous and hence $D_{f_D} = f(D)$ is connected in X.

□

Corollary 3.13. *Let E be a nonempty set in $X \times [0,1]$ with $E_r \supseteq E_t$ for $0 \leq t \leq r \leq S_E$. Then E_{S_E} is connected in (X,d) if and only if E is connected in $(X \times [0,1], \overline{d})$*

Proof. Let $D = \{(x, 1-\alpha) : (x,\alpha) \in E\}$. Then D is a nonempty set in $X \times [0,1]$ with $D_r \subseteq D_t$ for $f_D \leq t \leq r \leq 1$. Hence by Proposition 3.12, D_{f_D} is connected in (X,d) if and only if D is connected in $(X \times [0,1], \overline{d})$.

Define $f : (E, \overline{d}) \to (D, \overline{d})$ as follows: $f(x,\alpha) = (x, 1-\alpha)$ for $(x,\alpha) \in E$.

Observe that $E_{S_E} = D_{f_D}$, so to verify the desired result it suffices to show that D is connected in $(X \times [0,1], \overline{d})$ if and only if E is connected in $(X \times [0,1], \overline{d})$, which follows from that f is an isometry and $f(E) = D$.

The desired conclusion can also be proved in a similar manner as that of Proposition 3.12.

□

We use d_m to denote the Euclidean metric on $\mathbb{R}^m$. We also use $\mathbb{R}^m \times [0,1]$ to denote the metric space $(\mathbb{R}^m \times [0,1], \overline{d_m})$. $\mathbb{R}^1$ is written as $\mathbb{R}$.

Remark 3.14. Here we mention that $D_{S_D} = \emptyset$ is possible when D is connected in $X \times [0,1]$ and satisfies the assumption in Proposition 3.12. Let $X = \mathbb{R}$ and define $D \subset \mathbb{R} \times [0,1]$ by putting

$$D_\alpha = \begin{cases} \emptyset, & \alpha = 1, \\ (0, 1-\alpha], & \alpha \in [0,1). \end{cases}$$

Then D is a such example.

Similarly, $E_{f_E} = \emptyset$ is possible when E is connected in $X \times [0,1]$ and satisfies the assumption in Corollary 3.13.

Let $u \in F(X)$ and $0 \leq r \leq t \leq 1$. Define $\operatorname{end}_r^t u$ given by

$$\operatorname{end}_r^t u := \operatorname{end} u \cap ([u]_r \times [r,t]).$$

For simplicity, we write $\operatorname{end}_r^1 u$ as $\operatorname{end}_r u$. We can see that $\operatorname{end}_0 u = \operatorname{send} u$.

Clearly, for $u \in F_{USCG}(X)$ and $r \in (0,1]$, $\operatorname{end}_r u$ is a compact set in $X \times [0,1]$.

Corollary 3.15. *Let $u \in F(X)$.*

(i) For r, t with $0 \leq r \leq t \leq 1$, $\operatorname{end}_r^t u$ is connected in $X \times [0,1]$ if and only if $[u]_r$ is connected in X.

(ii) X is connected if and only if $\operatorname{end} u$ is connected in $X \times [0,1]$.

Proof. If $\operatorname{end}_r^t u \neq \emptyset$, then, by Proposition 3.12, the conclusion in (i) is true. Thus it suffices to consider the case when $\operatorname{end}_r^t u = \emptyset$. In this case, $[u]_r = \emptyset$, and so clearly the conclusion in (i) is true.

Note that $f_{\operatorname{end} u} = 0$ and $(\operatorname{end} u)_0 = X$. So (ii) follows immediately from Proposition 3.12.

□

The following Examples 3.16 and 3.17 give some connected sets in $\mathbb{R} \times [0,1]$. Proposition 3.12, Corollary 3.13 and Corollary 3.15 are used to show the connectedness of these sets.

Example 3.16. Let $D \subset \mathbb{R} \times [0,1]$ be defined by putting

$$D_\alpha = \begin{cases} [1-\alpha^2, 1] \cup [3, 4-\alpha^2], & \alpha \in (0.5, 1], \\ [0,4], & \alpha \in [0, 0.5]. \end{cases}$$

We can see that $D = A \cup B \cup C$, where

$$\begin{aligned} A &= [0,4] \times [0, 0.5], \\ B &= \bigcup_{\alpha \in [0.5,1]} [1-\alpha^2, 1] \times \{\alpha\}, \\ C &= \bigcup_{\alpha \in [0.5,1]} [3, 4-\alpha^2] \times \{\alpha\}. \end{aligned}$$

Clearly A is connected in $\mathbb{R} \times [0,1]$. By Corollary 3.13, B is connected in $\mathbb{R} \times [0,1]$. By Proposition 3.12, C is connected in $\mathbb{R} \times [0,1]$. $A \cap B = [1-0.5^2, 1] \times \{0.5\} \neq \emptyset$. $A \cap C = [3, 4-0.5^2] \times \{0.5\} \neq \emptyset$. Thus D is connected in $\mathbb{R} \times [0,1]$.

Here we mention that there is no $u \in F(\mathbb{R})$ satisfying $D = \operatorname{send} u$ because $D_1 \nsubseteq D_{0.9}$.

Example 3.17. Let $u \in F^1_{USC}(\mathbb{R})$ be defined by putting:

$$[u]_\alpha = \begin{cases} [0,1] \cup [3,4], & \alpha \in (0.6, 1], \\ [0,4], & \alpha \in [0, 0.6]. \end{cases}$$

We can see that $[0,1] \cup [3,4]$ is not connected, [0,4] and $\mathbb{R}$ are connected. So $u \in F^1_{USC}(\mathbb{R}) \setminus F_{USCCON}(\mathbb{R})$. By Corollary 3.15, $\operatorname{end} u$ is connected in $\mathbb{R} \times [0,1]$; $\operatorname{end}_r u$ is connected in $\mathbb{R} \times [0,1]$ if and only if $r \in [0, 0.6]$.

We say that a sequence $\{u_n, n = 1, 2, \ldots\}$ in $F_{USC}(\mathbb{R}^m)$ satisfies *connectedness condition* if for each $\varepsilon > 0$, there is a $\delta \in (0, \varepsilon]$ and $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$, $\operatorname{end}_\delta u_n$ is connected in $\mathbb{R}^m \times [0,1]$.

We will use the following conclusion.

Let (Y, ρ) be a metric space, $x, y \in Y$ and $W \subseteq Y$. Then

$$\begin{aligned} \rho(x, W) &= \inf_{z \in W} \rho(x, z) \\ &\leq \inf_{z \in W} \{\rho(x,y) + \rho(y,z)\} \\ &= \rho(x,y) + \rho(y, W). \end{aligned} \tag{2}$$

Theorem 3.18. *Let $u \in F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$, and let $\{u_n, n = 1, 2, \ldots\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ which satisfies the connectedness condition. Then $H_{\rm end}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

Proof. From Theorem 3.1, $H_{\rm end}(u_n, u) \to 0 \Rightarrow \lim^{(\Gamma)}_{n\to\infty} u_n = u$.

Now we show that $\lim^{(\Gamma)}_{n\to\infty} u_n = u \Rightarrow H_{\rm end}(u_n, u) \to 0$. We prove by contradiction. Assume that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ but $H_{\rm end}(u_n, u) \not\to 0$. Then $H^*(\operatorname{end} u, \operatorname{end} u_n) \not\to 0$ or $H^*(\operatorname{end} u_n, \operatorname{end} u) \not\to 0$. We split the proof into two cases.

Case (i) $H^*(\operatorname{end} u, \operatorname{end} u_n) \not\to 0$.

In this case, there is an $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $H^*(\operatorname{end} u, \operatorname{end} u_{n_k}) > \varepsilon$. Thus for each $k = 1, 2, \ldots$ there exists $(x_k, \alpha_k) \in \operatorname{end} u$ such that $\overline{d_m}((x_k, \alpha_k), \operatorname{end} u_{n_k}) > \varepsilon$. Note that for each $k = 1, 2, \ldots$ $\alpha_k > \varepsilon$. So $\{(x_k, \alpha_k),\ k = 1, 2, \ldots\} \subseteq \operatorname{end}_\varepsilon u$.

Since $\operatorname{end}_\varepsilon u$ is compact, there is a subsequence $\{(x_{k_l}, \alpha_{k_l})\}$ of $\{(x_k, \alpha_k)\}$ which converges to $(x, \alpha) \in \operatorname{end}_\varepsilon u \subset \operatorname{end} u$. Then there is a $L(\varepsilon)$ such that $\overline{d_m}((x, \alpha), (x_{k_l}, \alpha_{k_l})) < \varepsilon/2$ for all $l \geq L$. Hence, by (2), $\overline{d_m}((x, \alpha), \operatorname{end} u_{n_{k_l}}) \geq \overline{d_m}((x_{k_l}, \alpha_{k_l}), \operatorname{end} u_{n_{k_l}}) - \overline{d_m}((x, \alpha), (x_{k_l}, \alpha_{k_l})) > \varepsilon/2$. By Lemma 3.2 (i), this contradicts $(x, \alpha) \in \operatorname{end} u = \liminf_{n\to\infty} \operatorname{end} u_n$.

Case (ii) $H^*(\operatorname{end} u_n, \operatorname{end} u) \not\to 0$.

In this case, there is an $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$H^*(\operatorname{end} u_{n_k}, \operatorname{end} u) > \varepsilon. \tag{3}$$

Take $y \in \mathbb{R}^m$ with $u(y) > 0$. Set $\mu = u(y)$. Since $\{u_n\}$ satisfies the connectedness condition, there is a $\xi \in (0, \min\{\mu, \varepsilon\})$ and $N_1 \in \mathbb{N}$ such that $\operatorname{end}_\xi u_n$ is connected in $\mathbb{R}^m \times [0, 1]$ for all $n \geq N_1$.

Firstly we show the conclusions in the following clauses (I), (II) and (III).

(I) For each n_k, $k = 1, 2, \ldots$, there exists $(x_{n_k}, \alpha_{n_k}) \in \operatorname{end}_\xi u_{n_k}$ with $\overline{d_m}((x_{n_k}, \alpha_{n_k}), \operatorname{end}_\xi u) > \xi$.

(II) There is an $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$, there exists $(y_n, \beta_n) \in \operatorname{end}_\xi u_n$ with $\overline{d_m}((y_n, \beta_n), \operatorname{end}_\xi u) < \xi$.

(III) Set $N_3 := \max\{N_1, N_2\}$. For each $n_k \geq N_3$, there exists $(z_{n_k}, \gamma_{n_k}) \in \operatorname{end}_\xi u_{n_k}$ such that

$$\overline{d_m}((z_{n_k}, \gamma_{n_k}), \operatorname{end}_\xi u) = \xi. \tag{4}$$

To show (I), let $k \in \mathbb{N}$. From (3), there exists $(x_{n_k}, \alpha_{n_k}) \in \operatorname{end} u_{n_k}$ such that

$$\overline{d_m}((x_{n_k}, \alpha_{n_k}), \operatorname{end}_\xi u) \geq \overline{d_m}((x_{n_k}, \alpha_{n_k}), \operatorname{end} u) > \varepsilon > \xi.$$

Clearly $\alpha_{n_k} > \varepsilon$, and then $(x_{n_k}, \alpha_{n_k}) \in \operatorname{end}_\varepsilon u_{n_k} \subset \operatorname{end}_\xi u_{n_k}$. Thus (I) is true.

As $(y, \mu) \in \operatorname{end} u$ and $\operatorname{end} u = \liminf_{n\to\infty} \operatorname{end} u_n$, we can find a sequence $\{(y_n, \beta_n)\}$ satisfying that $(y_n, \beta_n) \in \operatorname{end} u_n$ for $n = 1, 2, \ldots$ and $\{(y_n, \beta_n)\}$ converges to (y, μ).

Hence there is a N_2 such that for all $n \geq N_2$, $\overline{d_m}((y_n, \beta_n), (y, \mu)) < \min\{\mu - \xi, \xi\}$. Let $n \in \mathbb{N}$ with $n \geq N_2$. Then $\beta_n > \xi$, and therefore $(y_n, \beta_n) \in \operatorname{end}_\xi u_n$. Note that $(y, \mu) \in \operatorname{end}_\xi u$. So $\overline{d_m}((y_n, \beta_n), \operatorname{end}_\xi u) \leq \overline{d_m}((y_n, \beta_n), (y, \mu)) < \xi$. Thus (II) is true.

To show (III), let $k \in \mathbb{N}$ with $n_k \geq N_3$. Define a function f_k from $(\operatorname{end}_\xi u_{n_k}, \overline{d_m})$ to $\mathbb{R}$ as follows:

$$f_k(z, \zeta) = \overline{d_m}((z, \zeta), \operatorname{end}_\xi u) \text{ for } (z, \zeta) \in \operatorname{end}_\xi u_{n_k}.$$

By (2), $|f_k(z, \zeta) - f_k(z', \zeta')| \leq \overline{d_m}((z, \zeta), (z', \zeta'))$ for $(z, \zeta), (z', \zeta')$ in $\operatorname{end}_\xi u_{n_k}$. Thus f_k is a continuous function on $\operatorname{end}_\xi u_{n_k}$.

Note that $\operatorname{end}_\xi u_{n_k}$ is a connected set in $\mathbb{R}^m$. Thus $f_k(\operatorname{end}_\xi u_{n_k})$ is a connected set in $\mathbb{R}$; that is, $f_k(\operatorname{end}_\xi u_{n_k})$ is an interval. Combined this fact with the above clauses (I) and (II), we obtain that there exists $(z_{n_k}, \gamma_{n_k}) \in \operatorname{end}_\xi u_{n_k}$ with $\overline{d_m}((z_{n_k}, \gamma_{n_k}), \operatorname{end}_\xi u) = \xi$. Thus (III) is true.

Now using (III), we can obtain a contradiction. From (4) and the compactness of $\operatorname{end}_\xi u$, the set $\{(z_{n_k}, \gamma_{n_k}), n_k \geq N_3\}$ is bounded in $\mathbb{R}^m \times [0, 1]$, and thus $\{(z_{n_k}, \gamma_{n_k}), n_k \geq N_3\}$ has a cluster point (z, γ). So $(z, \gamma) \in \limsup_{n\to\infty} \operatorname{end} u_n = \operatorname{end} u$. As $(z_{n_k}, \gamma_{n_k}) \in \operatorname{end}_\xi u_{n_k}$, we have that $\gamma_{n_k} \geq \xi$ and therefore $\gamma \geq \xi$. Thus $(z, \gamma) \in \operatorname{end} u \cap (X \times [\xi, 1]) = \operatorname{end}_\xi u$. But, by (4), $\overline{d_m}((z, \gamma), \operatorname{end}_\xi u) = \xi$, which is a contradiction.

$\square$

Corollary 3.19. *Let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$ and for $n = 1, 2, \ldots$, let u_n be a fuzzy set in $F_{USCCON}(\mathbb{R}^m)$. Then $H_{\text{end}}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

Proof. Let $\{u_n : n \in \mathbb{N}\}$ is a sequence in $F_{USCCON}(\mathbb{R}^m)$. Then, by clause (i) of Corollary 3.15, for each $n \in \mathbb{N}$ and $r \in (0, 1]$, $\operatorname{end}_r u_n$ is a connected set in $\mathbb{R}^m \times [0, 1]$. Hence $\{u_n : n \in \mathbb{N}\}$ satisfies the connectedness condition. Thus the desired result follows immediately from Theorem 3.18.

$\square$

Here we mention that for $u \in F^1_{USCG}(\mathbb{R})$ and a sequence $\{u_n\}$ in $F^1_{USC}(\mathbb{R})$ which satisfies the connectedness condition, $H(u_n, u) \to 0$ does not imply that there is an N such that for each $n \geq N$, $u_n \in F_{USCCON}(\mathbb{R})$.

The following Example 3.20 is a such example, which shows that there exists a $u \in F^1_{USCG}(\mathbb{R})$ and a sequence $\{u_n\}$ in $F^1_{USC}(\mathbb{R})$ such that
(i) $H(u_n, u) \to 0$,
(ii) $\{u_n\}$ satisfies the connectedness condition, and
(iii) for each $n = 1, 2, \ldots$, $u_n \notin F_{USCCON}(\mathbb{R})$.

Example 3.20. For $n = 1, 2, \ldots$, let u_n be the fuzzy set u given in Example 3.17; that is, $u_n = u$ is a fuzzy set in $F^1_{USCG}(\mathbb{R})$ defined by putting:

$$[u]_\alpha = \begin{cases} [0,1] \cup [3,4], & \alpha \in (0.6, 1], \\ [0,4], & \alpha \in [0, 0.6]. \end{cases}$$

We have the following conclusions:
(i) $H(u_n, u) \to 0$ since $u_n = u$ for $n = 1, 2, \ldots$;
(ii) $\{u_n\}$ satisfies the connectedness condition because $\operatorname{end}_r u$ is connected in $\mathbb{R} \times [0,1]$ when $r \in [0, 0.6]$ (see Example 3.17);
(iii) for each $n = 1, 2, \ldots$, $u_n \notin F_{USCCON}(\mathbb{R})$, as $u \notin F_{USCCON}(\mathbb{R})$ (see Example 3.17).

Remark 3.21. Theorem 9.2 in [5], which is Theorem 3.10 in this paper, is a corollary of Corollary 3.19 since $F_{USCGCON}(\mathbb{R}^m) \subseteq F_{USCCON}(\mathbb{R}^m)$.

Theorem 3.7 is a corollary of Corollary 3.19, as $F^1_{USCG}(\mathbb{R}^m) \subseteq F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$ and $F^1_{USCCON}(\mathbb{R}^m) \subseteq F_{USCCON}(\mathbb{R}^m)$.

Corollary 3.19 is a corollary of Theorem 3.18.

Theorem 3.18 improves Theorem 9.2 in [5] and Theorem 3.7.

Now we give an improvement of Theorem 3.18.

Let $u \in F(X)$. Define $S(u) := \sup\{u(x) : x \in X\}$. Clearly $[u]_{S_u} = \emptyset$ is possible.

Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ and $\{u_{n_k}\}$ a subsequence of $\{u_n\}$. If there is a $u \in F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$ such that $u = \lim^{(\Gamma)}_{n\to\infty} u_n$, then we can define

$$A^1_{n_k} := \{\xi > 0 : \text{ for each } n_k,\ H^*(\operatorname{end} u, \operatorname{end} u_{n_k}) > \xi\},$$
$$A^2_{n_k} := \{\xi > 0 : \text{ for each } n_k,\ H^*(\operatorname{end} u_{n_k}, \operatorname{end} u) > \xi\}.$$

Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ and let u be a fuzzy set in $F_{USC}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$. We call the pair $\{u_n\}$, u is a *weak connectedness compact pair* if one of the following (i) and (ii) holds:
(i) $\lim^{(\Gamma)}_{n\to\infty} u_n$ does not exist, or $\lim^{(\Gamma)}_{n\to\infty} u_n$ exists but $u = \lim^{(\Gamma)}_{n\to\infty} u_n$ does not hold;
(ii) $u = \lim^{(\Gamma)}_{n\to\infty} u_n$ and (ii-1) and (ii-2) are true.

(ii-1) for each $\{u_{n_k}\}$ with $A^1_{n_k} \neq \emptyset$, there exists a $\xi \in A^1_{n_k}$ such that $\mathrm{end}_\xi\, u$ is compact.

(ii-2) for each $\{u_{n_k}\}$ with $A^2_{n_k} \neq \emptyset$, there exists a $\xi \in A^2_{n_k}$ and an $N(\xi) \in \mathbb{N}$ such that $\xi < S(u)$, $\mathrm{end}_\xi\, u$ is compact, and $\mathrm{end}_\xi\, u_{n_k}$ is connected in $\mathbb{R}^m \times [0,1]$ for all $n_k \geq N$.

Theorem 3.22. *Let $u \in F_{USCG}(\mathbb{R}^m)\setminus\{\emptyset_{F(\mathbb{R}^m)}\}$, and let $\{u_n, n = 1, 2, \ldots\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$. If the pair $\{u_n\}$, u is a weak connectedness compact pair, then $H_{\mathrm{end}}(u_n, u) \to 0$ as $n \to \infty$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.*

Proof. The proof is similar to that of Theorem 3.18.

From Theorem 3.1, $H_{\mathrm{end}}(u_n, u) \to 0 \Rightarrow \lim^{(\Gamma)}_{n\to\infty} u_n = u$.

Now we show that $\lim^{(\Gamma)}_{n\to\infty} u_n = u \Rightarrow H_{\mathrm{end}}(u_n, u) \to 0$. We prove by contradiction. Assume that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ but $H_{\mathrm{end}}(u_n, u) \not\to 0$. Then $H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) \not\to 0$ or $H^*(\mathrm{end}\, u_n, \mathrm{end}\, u) \not\to 0$. We split the proof into two cases.

Case (i) $H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) \not\to 0$.

In this case, $A^1_{n_k} \neq \emptyset$. Since the pair $\{u_n\}$, u is a weak connectedness compact pair, then there is a $\xi \in A^1_{n_k}$ with $\mathrm{end}_\xi\, u$ is compact. So we can prove that there is a contradiction in a similar manner to that in the case (i) of the proof of Theorem 3.18.

Case (ii) $H^*(\mathrm{end}\, u_n, \mathrm{end}\, u) \not\to 0$.

In this case, $A^2_{n_k} \neq \emptyset$. Since the pair $\{u_n\}$, u is a weak connectedness compact pair, there is a $\xi \in A^2_{n_k}$ and $N_1 \in \mathbb{N}$ such that $\xi < S(u)$, $\mathrm{end}_\xi\, u$ is compact, and for all $n_k \geq N_1$, $\mathrm{end}_\xi\, u_{n_k}$ is connected in $\mathbb{R}^m \times [0,1]$.

Firstly we show the conclusions in the following clauses (I), (II) and (III).

(I) For each n_k, $k = 1, 2, \ldots$, there exists $(x_{n_k}, \alpha_{n_k}) \in \mathrm{end}_\xi\, u_{n_k}$ with $\overline{d_m}((x_{n_k}, \alpha_{n_k}), \mathrm{end}_\xi\, u) > \xi$.

(II) There is an $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$, there exists $(y_n, \beta_n) \in \mathrm{end}_\xi\, u_n$ with $\overline{d_m}((y_n, \beta_n), \mathrm{end}_\xi\, u) < \xi$.

(III) Set $N_3 := \max\{N_1, N_2\}$. For each $n_k \geq N_3$, there exists $(z_{n_k}, \gamma_{n_k}) \in \operatorname{end}_\xi u_{n_k}$ such that

$$\overline{d_m}((z_{n_k}, \gamma_{n_k}), \operatorname{end}_\xi u) = \xi. \tag{5}$$

Note that $\xi \in A^2_{n_k}$; that is, for each u_{n_k}, $k = 1, 2, \ldots$

$$H^*(\operatorname{end} u_{n_k}, \operatorname{end} u) > \xi. \tag{6}$$

Let $k \in \mathbb{N}$. From (6), there exists $(x_{n_k}, \alpha_{n_k}) \in \operatorname{end} u_{n_k}$ such that

$$\overline{d_m}((x_{n_k}, \alpha_{n_k}), \operatorname{end}_\xi u) \geq \overline{d_m}((x_{n_k}, \alpha_{n_k}), \operatorname{end} u) > \xi.$$

Clearly $\alpha_{n_k} > \xi$, and then $(x_{n_k}, \alpha_{n_k}) \in \operatorname{end}_\xi u_{n_k}$. Thus (I) is true.

Since $\xi < S(u)$, we can take $y \in \mathbb{R}^m$ with $u(y) > \xi > 0$. Set $u(y) = \mu$. As $(y, \mu) \in \operatorname{end} u$ and $\operatorname{end} u = \liminf_{n\to\infty} \operatorname{end} u_n$, we can find a sequence $\{(y_n, \beta_n)\}$ satisfying that $(y_n, \beta_n) \in \operatorname{end} u_n$ for $n = 1, 2, \ldots$ and $\{(y_n, \beta_n)\}$ converges to (y, μ).

Hence there is a N_2 such that for all $n \geq N_2$, $\overline{d_m}((y_n, \beta_n), (y, \mu)) < \min\{\mu - \xi, \xi\}$. Let $n \in \mathbb{N}$ with $n \geq N_2$. Then $\beta_n > \xi$, and therefore $(y_n, \beta_n) \in \operatorname{end}_\xi u_n$. Note that $(y, \mu) \in \operatorname{end}_\xi u$. So $\overline{d_m}((y_n, \beta_n), \operatorname{end}_\xi u) \leq \overline{d_m}((y_n, \beta_n), (y, \mu)) < \xi$. Thus (II) is true.

(III) follows from (I), (II) and the connectedness of $\operatorname{end}_\xi u_{n_k}$ when $n_k \geq N_3$. The proof of (III) is the same as that of the clause (III) in the proof of Theorem 3.18.

Now using (III) and the compactness of $\operatorname{end}_\xi u$, we can have a contradiction. The proof is the same as the counterpart in the proof of Theorem 3.18.

□

Example 3.23. Let u be a fuzzy set in $F^1_{USC}(\mathbb{R}) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$ defined by putting:

$$[u]_\alpha = \begin{cases} \{1\}, & \alpha \in (0.6, 1], \\ (-\infty, -1] \cup [1, +\infty) & \alpha \in [0, 0.6]. \end{cases}$$

For $n = 1, 2, \ldots$, let $u_n = u$. We have the following conclusions:
(i) $H(u_n, u) \to 0$ since $u_n = u$ for $n = 1, 2, \ldots$;
(ii) the pair $\{u_n\}$, u is a weak connectedness compact pair;
(iii) $\{u_n\}$ does not satisfy the connectedness condition because $\operatorname{end}_r u$ is not connected in $\mathbb{R} \times [0, 1]$ when $r \in [0, 0.6]$. Clearly each subsequence $\{u_{n_k}\}$ of $\{u_n\}$ does not satisfy the connectedness condition;
(iv) $u \notin F_{USCG}(\mathbb{R})$.

Remark 3.24. Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ satisfying the connectedness condition and let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_{F(\mathbb{R}^m)}\}$. Then clearly the pair $\{u_n\}$, u is a weak connectedness compact pair. So Theorem 3.18 is a corollary of Theorem 3.22. Theorem 3.22 is an improvement of Theorem 3.18.

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